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Factorization and unitarity in superstring theory

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Abstract: The overall coefficient of the two-loop 4-particle amplitude in superstring theory is determined by making use of the factorization and unitarity. To accomplish this we computed in detail all the relevant tree and one-loop amplitudes involved and determined their overall coefficients in a consistent way.

Keywords: Field Theories in Higher Dimensions, Superstrings and Heterotic Strings.

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Contents

1. Introduction

Explicit result for higher loop amplitudes in superstring is quite rare. To our knowledge the only explicitly known higher loop (≥ 2) non-vanishing amplitude is the four-particle amplitude in superstring theory, firstly obtained in [1] and later re-obtained in [2−4] in an explicitly gauge independent way, following the works of D' Hoker and Phong [5−10] on two loop measure of superstring theory. This result was also computed in a super-Poincare covariant way in [11]. Recently D' Hoker and Phong [12, 13] also gave a measure for three loop superstring theory. It remains to see if this can be used to do explicit three loop computations in superstring theory. For another promising approach of covariant calculation of superstring amplitudes we refer the reader to Berkovits' review [14].

Due to the rareness of explicit results, it is natural to study the known result in depth. The old result was cast into an explicit modular invariant form [15] and used in [16] to prove the vanishing of the R^4 correction [17, 18]. It has also been proved in [19] that the results obtained in [1−3] are equivalent. Another goal we have in mind is to make connection with known results from field theory in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [20, 21]. It seems natural to compute the precise overall coefficient of the two-loop 4-particle amplitude.

This seems a trivial problem, but in fact it turns out to be quite involved. One could try to use factorization and unitarity to fix the overall coefficient for the four-particle amplitude. In a previous paper [22] we have studied in detail the factorization of the two-loop 4-particle amplitude in superstring theory.[1] When we use this result to determine the coefficient of the two-loop 4-particle amplitude, we found that we need the precise overall factor for other one-loop amplitudes involved. Due to the incomplete results in literature (and a fear of wrongly quoting other's results), we therefore computed all the relevant amplitudes in a consistent way and fixed all the overall factors by either using factorization or unitarity. In this way the coefficient of the 2-loop 4-particle amplitude is determined exactly. This paper is organized as follows:

In the next section we recall all the vertex operators needed in this paper, following the covariant quantization of the Neveu-Schwarz-Ramond theory by Friedan, Martinec and Shenker [25] and Knizhnik [26]. In section 3 we gave all the results for tree amplitudes needed with their overall coefficients. We omit all the computations as this becomes standard exercises in superstring theory. We don't claim any originality for these amplitudes although some amplitudes with massive tensor may be new. The one-loop amplitudes are collected in section 4. Starting from the 4-particle amplitude we obtained the 3-particle (one massive tensor) and 2-particle (both massive tensor) amplitudes by factorization. The overall coefficient is determined by using unitarity (which is the only place we used unitarity relation). The summation over the intermediate states is quite involved and 2 appendices are devoted to the proof of an equation (eq. (4.17)). In section 5 we combine the results of the previous sections and the result of [22] to determined the overall coefficient of the 2-loop 4-particle amplitude.

During the writing of this paper we received the paper of D'Hoker, Gutperle and Phong [27] which also determined the precise overall coefficient of the 2-loop 4-particle amplitude. The method used for the final determination of the 2-loop coefficient is the same. Nevertheless they used the result of S-duality for the determination of the 1-loop coefficient. We found complete agreement with their result although the factorization is performed by using hyperelliptic language.

2. A review of the vertex operators

First let us set our notations for the vertex operators used. For all our calculations we use the covariant emission vertices constructed by Friedan, Martinec and Shenker [25], and by Knizhnik [26]. Let us briefly review their construction. Most of the time we will present the result for the left-moving (or holomorphic) part, as the formalism for the right-moving (or anti-holomorphic) part is the same. So it is useful to introduce a set of notations to separate the two parts. As we will use only three different vertex operators in this paper, we use the subscripts B and F to denote the bosonic and fermionic massless vertex operators from the Neveu-Schwarz sector and Ramond sector respectively. The level one (or the first) massive vertex operator from the Neveu-Schwarz sector is denoted by a subscript M.

[1] See also [23, 24] for early works on two-loop factorization.

Subscripts with an additional ˜ denote the right-moving part when we need the complete amplitude in superstring theory.

By using these notations, the vertex operator for the massless NS-$\widetilde{NS}$ tensor is:

$$\mathcal{V}^{(-1,-1)}_{B\widetilde{B}}(z,\bar{z},k,\epsilon,\tilde{\epsilon}) = \mathcal{V}^{(-1)}_{B}(z,k,\epsilon)\mathcal{V}^{(-1)}_{\widetilde{B}}(\bar{z},k,\tilde{\epsilon})\,, \tag{2.1}$$

where

$$\mathcal{V}^{(-1)}_{B}(z,k,\epsilon) = g_c\,\epsilon\cdot\psi(z)\,\mathrm{e}^{-\phi(z)}\,\mathrm{e}^{ik\cdot X(z,\bar{z})}\,, \tag{2.2}$$

$$\mathcal{V}^{(-1)}_{\widetilde{B}}(\bar{z},k,\tilde{\epsilon}) = \tilde{\epsilon}\cdot\tilde{\psi}(\bar{z})\,\mathrm{e}^{-\tilde{\phi}(\bar{z})}\,. \tag{2.3}$$

Here in the above, we have written the polarization tensor in a factorized form: $\epsilon_{\mu\nu} = \epsilon_\mu\tilde{\epsilon}_\nu$. By convention we absorbed the overall constant g_c and the exponential factor $\mathrm{e}^{ik\cdot X(z,\bar{z})}$ into the left-moving vertex operator $\mathcal{V}_B$. One may also split $X(z,\bar{z})$ into a left-moving part and a right-moving part, but we will not do this in this paper as this is not essential for our purpose.

We note that the vertex operator given in eq. (2.2) carries a ghost charge of -1. We will also need the physical equivalent vertex operator which carries no ghost charge. It is given as follows:

$$\mathcal{V}^{(0)}_{B}(z,k,\epsilon) = -g_c(\epsilon\cdot\partial X(z) + ik\cdot\psi(z)\epsilon\cdot\psi(z))\,\mathrm{e}^{ik\cdot X(z)}\,. \tag{2.4}$$

This vertex operator is obtained from the ghost charge -1 operator of eq. (2.2) by using the "picture-raising" operator $Z(y)$:

$$Z(y) = \{Q, 2\xi(y)\} = -P\cdot\psi\,\mathrm{e}^{\phi} + \cdots\,, \tag{2.5}$$

$$\begin{aligned}\mathcal{V}^{(0)}_{B}(z,k,\epsilon) &= \,:Z(z)\mathcal{V}^{(-1)}_{B}(z,k,\epsilon):\\ &= \frac{1}{2\pi i}\oint_z \frac{\mathrm{d}y}{y-z}\,Z(y)\mathcal{V}^{(-1)}_{B}(z,k,\epsilon)\,,\end{aligned} \tag{2.6}$$

modulo spurious operators.[2] All other vertex operators of different ghost charge can be related in the same way. They are physically equivalent [25].

The second vertex operator (left-moving part only) is the massive tensor from the Neveu-Schwarz sector [28 – 30]:

$$\mathcal{V}^{(-1)}_{M} = g_M\Big\{\alpha_{\mu\nu\rho}i\psi^\mu(z)\psi^\nu(z)\psi^\rho(z) + \sigma_{\mu\nu}\partial X^\mu(z)\psi^\nu(z) - \sigma_\mu i\partial\psi^\mu(z)\Big\}\mathrm{e}^{-\phi(z)}\,\mathrm{e}^{ik\cdot X(z)}\,, \tag{2.7}$$

which is in the (-1)-picture (or ghost charge -1) and

$$\begin{aligned}\mathcal{V}^{(0)}_{M} = -g_M\Bigg\{&\alpha_{[\mu\nu\rho]}\left[3i\partial X^\mu\psi^\nu\psi^\rho + \frac{\alpha'}{2}\,k\cdot\psi\psi^\mu\psi^\nu\psi^\rho\right] +\\ &+\left(\frac{2}{\alpha'}\right)^{1/2}\sigma_{\mu\nu}\left[\partial X^\mu\partial X^\nu + \frac{\alpha'}{2}\,(\partial\psi^\mu\psi^\nu + ik\cdot\psi\partial X^\mu\psi^\nu)\right] -\\ &-\sigma_\mu\left[i\partial^2 X - \frac{\alpha'}{2}\,k\cdot\psi\partial\psi^\mu\right]\Bigg\}\mathrm{e}^{ik\cdot X}\,,\end{aligned} \tag{2.8}$$

[2] Exactly we have:

$$:Z(z)\psi^\mu(z)\,\mathrm{e}^{-\phi(z)+ik\cdot X(z)} := (ik^\mu(\eta\xi + \partial\phi) - (\partial X^\mu + ik\cdot\psi\psi^\mu))\,\mathrm{e}^{ik\cdot X(z)}\,.$$

which is in the 0-picture and the dependence on the dimensional scale α' is restored for $\mathcal{V}_{M}^{(0)}$. The mass-shell condition is $k^2 = -\frac{4}{\alpha'}$. Here $\alpha_{\mu\nu\rho}$ and $\sigma_{\mu\nu}$ are the polarization tensors which satisfies the following normalization conditions:

$$\alpha_{\mu\nu\rho}(k)\alpha^{\mu\nu\rho}(-k) = -\frac{1}{6}\,, \qquad \sigma_{\mu\nu}(k)\sigma^{\mu\nu}(-k) = 1\,. \tag{2.9}$$

which are given in [22]. The state represented by σ_μ is null and it will not appear in the physical amplitude.

The last vertex operator is the massless fermion from the Ramond sector:

$$V_F^{(-\frac{1}{2})} = g_{_F}\mathrm{e}^{-\phi/2}u^\alpha S_\alpha\,\mathrm{e}^{-\tilde{\phi}(\bar{z})}\,\mathrm{e}^{ik\cdot X(z)}\,, \tag{2.10}$$

which is in the $(-\frac{1}{2})$-picture and u is a Majorana-Weyl spinor in 10-dimensional space-time. We will not need the expression in the $\frac{1}{2}$-picture which we give it here

$$V_F^{(\frac{1}{2})} = g_{_F}u^\alpha\left\{\mathrm{e}^{\phi/2}[\partial X^\mu + \frac{i}{4}\,k\cdot\psi\psi^\mu]\gamma_{\mu\alpha\beta}\,S^\beta + \frac{1}{2}\,\mathrm{e}^{3\phi/2}\eta b S_\alpha\right\}\mathrm{e}^{ik\cdot X}\,, \tag{2.11}$$

just for completeness (see [25] for details).

As a last note, our convention for the S-matrix is:

$$S(1,\cdots,N) = \delta(1,\cdots,N) + (2\pi)^D\delta^D(k_1+\cdots+k_N)\,i\mathcal{A}_N(1,\cdots,N)\,, \tag{2.12}$$

where all momenta are incoming and $D = 10$ for superstring theory. All the formulas are given in terms of $\mathcal{A}_N$ and the momentum conservation is implicit in it.

3. The tree-level amplitudes and their factorization in superstring theory

3.1 The massless boson amplitudes

A general tree-level n-particle (assuming to be all massless NS bosons) amplitude is computed as follows:

$$\begin{aligned} i\mathcal{A}_n(k_i,\epsilon_i) = \int\prod_{i=4}^n \mathrm{d}^2z_i\Big\langle &[c\mathcal{V}_B^{(-1)}](z_1,k_1,\epsilon_1)[c\mathcal{V}_B^{(0)}](z_2,k_2,\epsilon_2)\times \qquad (3.1)\\ &\times[\mathcal{V}_B^{(-1)}](z_3,k_3,\epsilon_3)\prod_{i=4}^n[\mathcal{V}_B^{(0)}](z_i,k_i,\epsilon_i)(\text{right-moving part})\Big\rangle\,, \end{aligned}$$

by fixing the first three insertion points of the vertex operators and integrating the rest insertion points [31]. To obtain a non-trivial amplitude ($n \geq 4$) we do need the right-moving part explicitly. The computation is straightforward but quite tedious.

The results for $n = 3$ and $n = 4$ are well-known and are given as follows [31]:

$$\mathcal{A}_3(k_i,\epsilon_i,\tilde{\epsilon}_i) = \frac{8\,\pi\,g_c}{\alpha'}\,K_3(k_i,\epsilon_i)K_3(k_i,\tilde{\epsilon}_i)\,, \tag{3.2}$$

$$\begin{aligned}\mathcal{A}_4(k_i,\epsilon_i,\tilde{\epsilon}_i) &= c\times\frac{-\kappa^2(\alpha')^3}{4}\,K(k_i,\epsilon_i)K(k_i,\tilde{\epsilon}_i)\times\\ &\quad\times\frac{\Gamma(-\frac{\alpha's}{4})\Gamma(-\frac{\alpha't}{4})\Gamma(-\frac{\alpha'u}{4})}{\Gamma(1+\frac{\alpha's}{4})\Gamma(1+\frac{\alpha't}{4})\Gamma(1+\frac{\alpha'u}{4})}\,,\end{aligned} \tag{3.3}$$

where the various kinematic factors are given as follows:

$$\begin{aligned}
K_3(k_i,\epsilon_i) &= \left(\frac{\alpha'}{2}\right)^{1/2}(\epsilon_1\cdot\epsilon_2\epsilon_3\cdot k_1+\epsilon_1\cdot\epsilon_3\epsilon_2\cdot k_3+\epsilon_2\cdot\epsilon_3\epsilon_1\cdot k_2)\,,\\
K(k_i,\epsilon_i) &= -\frac{(\alpha')^2}{16}\Big[ut\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4+st\epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4+us\epsilon_1\cdot\epsilon_4\epsilon_2\cdot\epsilon_3\Big]+\\
&\quad+\frac{t(\alpha')^2}{8}\Big[\epsilon_1\cdot\epsilon_2\epsilon_3\cdot k_1\epsilon_4\cdot k_2+\epsilon_1\cdot\epsilon_3\epsilon_2\cdot k_1\epsilon_4\cdot k_3+\\
&\qquad\qquad+\epsilon_2\cdot\epsilon_4\epsilon_1\cdot k_2\epsilon_3\cdot k_4+\epsilon_3\cdot\epsilon_4\epsilon_2\cdot k_4\epsilon_1\cdot k_3\Big]+\\
&\quad+\frac{u(\alpha')^2}{8}\Big[\epsilon_1\cdot\epsilon_2\epsilon_3\cdot k_2\epsilon_4\cdot k_1+\epsilon_1\cdot\epsilon_4\epsilon_2\cdot k_1\epsilon_3\cdot k_4+\\
&\qquad\qquad+\epsilon_2\cdot\epsilon_3\epsilon_1\cdot k_2\epsilon_4\cdot k_3+\epsilon_3\cdot\epsilon_4\epsilon_1\cdot k_4\epsilon_2\cdot k_3\Big]+\\
&\quad+\frac{s(\alpha')^2}{8}\Big[\epsilon_1\cdot\epsilon_3\epsilon_2\cdot k_3\epsilon_4\cdot k_1+\epsilon_2\cdot\epsilon_4\epsilon_1\cdot k_4\epsilon_3\cdot k_2+\\
&\qquad\qquad+\epsilon_2\cdot\epsilon_3\epsilon_1\cdot k_3\epsilon_4\cdot k_2+\epsilon_1\cdot\epsilon_4\epsilon_2\cdot k_4\epsilon_3\cdot k_1\Big]\,,
\end{aligned}\tag{3.4}$$

which are dimensionless. Here $K(k_i,\epsilon_i)$ is the standard (left-moving part) kinematic factor from the tree, one-loop and two-loop computations in superstring theory [32, 1, 3]. In our convention all the kinematic factors are dimensionless (and they agree with the previous ones after setting $\alpha'=2$). The right-moving part kinematic factors are obtained from the corresponding left-moving part kinematic factors by the simple substitution $\epsilon_i\to\tilde\epsilon_i$. For example we have:

$$K_3(k_i,\tilde\epsilon_i)=\left(\frac{\alpha'}{2}\right)^{1/2}(\tilde\epsilon_1\cdot\tilde\epsilon_2\tilde\epsilon_3\cdot k_1+\tilde\epsilon_1\cdot\tilde\epsilon_3\tilde\epsilon_2\cdot k_3+\tilde\epsilon_2\cdot\tilde\epsilon_3\tilde\epsilon_1\cdot k_2)\,.\tag{3.5}$$

This rule will apply for all later formulas involving right-moving part contribution if we don't explicitly say an alternative.

The constant c appearing in (3.3) can be proved to be equal 1 by using factorization property of the amplitude. In order to do this, we must use the following result for the summation over intermediate states:

$$\sum_{\epsilon(k)}K_3(k_1,\epsilon_1;k_2,\epsilon_2;k,\epsilon(k))\,K_3(k_3,\epsilon_3;k_4,\epsilon_4;-k,\epsilon(-k))=-\frac{4}{u\alpha'}\,K(k_i,\epsilon_i)|_{s=0,t=-u}\,,\tag{3.6}$$

which can be easily proved by using the following formula:

$$\sum_{\epsilon(k)}\epsilon_\mu(k)\,\epsilon_\nu(-k)=\sum_{\epsilon(k)}\epsilon_\mu(k)\,\epsilon^*_\nu(k)=\eta_{\mu\nu}-\frac{1}{2k\cdot p}\,(k_\mu p_\nu+k_\nu p_\mu)\,,\tag{3.7}$$

where p is a reference momentum and can be chosen as $(p^\mu)=(k^0,-k^i)$.

3.2 The two massless boson and one massive tensor vertex

The left-moving part vertex for two massless boson and one massive tensor is:

$$\begin{aligned}
i\mathcal{A}_{2BM} &= \Big\langle[c\mathcal{V}_B^{(-1)}](z_1,k_1)[c\mathcal{V}_B^{(0)}](z_2,k_2)[c\mathcal{V}_M^{(-1)}](z,k)\Big\rangle\\
&= ig_{_M}\,K_M(k_1,\epsilon_1;k_2,\epsilon_2;k,\alpha,\sigma)\,,
\end{aligned}\tag{3.8}$$

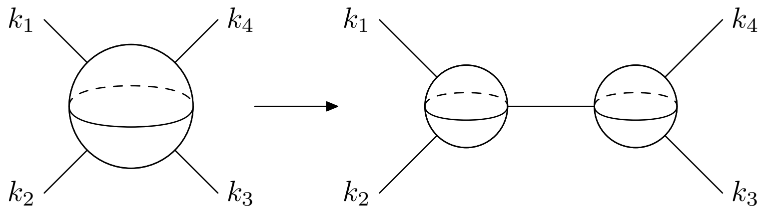

Figure 1: The factorization of the 4-particle tree amplitude. The intermediate state is a massive tensor.

where the kinematic factor K_M is:

$$K_M = -6\left(\frac{\alpha'}{2}\right)^{1/2}\alpha_{\mu\nu\rho}k_1^\mu\epsilon_1^\nu\epsilon_2^\rho + \sigma_{\mu\nu}[\epsilon_1^\mu\epsilon_2^\nu + \frac{\alpha'}{2}\left(k_1^\mu k_1^\nu \epsilon_1\cdot\epsilon_2 - k_1^\mu\epsilon_1^\nu\epsilon_2\cdot k_1 + k_1^\mu\epsilon_2^\nu\epsilon_1\cdot k_2\right)]\,. \quad (3.9)$$

As we said before, there is no contribution from the σ_μ term of the vertex operator as it gives spurious physical states.

By combining the left-moving part vertex with right-moving part vertex, we can use the factorization property of the 4-particle amplitude for $s \to \frac{4}{\alpha'}$ to obtain the overall coefficient g_M as we did in the last subsection. Here we must use the following formula for the summation over intermediate massive states:

$$\begin{aligned}\sum_{\alpha(k),\sigma(k)} & K_M(k_1,\epsilon_1;k_2,\epsilon_2;k,\alpha(k),\sigma(k))\times \\ & \times K_M(k_3,\epsilon_3;k_4,\epsilon_4;-k,\alpha(-k),\sigma(-k)) \;=\; K(k_i,\epsilon_i)\big|_{s=\frac{4}{\alpha'}}\,. \end{aligned} \quad (3.10)$$

This is proved in [22].

By using eq. (3.10) and the factorization of the 4-particle amplitude into two 3-particle (one is massive) amplitude as shown in figure 1, we get

$$g_M = \frac{4\kappa}{\alpha'} = \frac{8\pi g_c}{\alpha'}\,. \quad (3.11)$$

by choosing it to be positive.

3.3 Tree amplitudes with fermions or R-R tensors

In this subsection we only list all the required results. They are needed to prove eq. (4.17). It can be skipped if one is only interested in the result for the overall coefficient.

First we have two 3-particle vertices. These are given as follows:

- Vertex $(NS,\widetilde{NS}) \to (R,\widetilde{NS}) + (R,\widetilde{NS})$:

$$\mathcal{A}_F(k_1,u_1,\tilde\epsilon_1;k_2,u_2,\tilde\epsilon_2;k_3,\epsilon_3,\tilde\epsilon_3) = -g_F\sqrt{\frac{\alpha'}{2}}\bar u_1 \not\epsilon_3 u_2\, K_3(k_i,\tilde\epsilon_i)\,, \quad (3.12)$$

with

$$g_F = \frac{4\,\pi\, g_c}{\alpha'}\,. \quad (3.13)$$

- Vertex $(NS, \widetilde{NS}) + (R, \widetilde{R}) \to (R, \widetilde{R})$:

$$\mathcal{A}_R(k_1, u_1, \tilde{u}_1; k_2, u_2, \tilde{u}_2; k_3, \epsilon_3, \tilde{\epsilon}_3) \;=\; g_R K_R \tilde{K}_R \tag{3.14}$$

where

$$g_R \;=\; \frac{2\pi\, g_c}{\alpha'}\,, \tag{3.15}$$

$$K_R \;=\; \sqrt{\frac{\alpha'}{2}}\, \bar{u}_1 \Gamma^\mu u_2 \epsilon_{3\mu}\,. \tag{3.16}$$

For the 4-particle amplitude, the massless 2 fermion and 2 boson $((R, \widetilde{NS}) + (NS, \widetilde{NS}) \to (R, \widetilde{NS}) + (NS, \widetilde{NS}))$ amplitude is:

$$\mathcal{A}_{FBFB} = g_{2F}\, K_{FBFB}\, \tilde{K}(k_i, \tilde{\epsilon}_i)\, \frac{\Gamma(-\frac{\alpha' s}{4})\Gamma(-\frac{\alpha' t}{4})\Gamma(-\frac{\alpha' u}{4})}{\Gamma(1+\frac{\alpha' s}{4})\Gamma(1+\frac{\alpha' t}{4})\Gamma(1+\frac{\alpha' u}{4})}\,, \tag{3.17}$$

with

$$g_{2F} \;=\; \frac{2\pi^2\, g_c^2}{\alpha'}\,. \tag{3.18}$$

Here the kinematic factor K_{FBFB} is:

$$\begin{aligned} K_{FBFB} \;=\; & -\frac{t(\alpha')^2}{4}\left[\epsilon_4\cdot k_3 \bar{u}_1 \not\epsilon_2 u_3 + \frac{1}{2} k_{4\mu}\epsilon_{4\nu}\bar{u}_1\not\epsilon_2 \Gamma^{[\mu\nu]} u_3\right] - \\ & -\frac{s(\alpha')^2}{4}\left[\epsilon_4\cdot k_1 \bar{u}_1 \not\epsilon_2 u_3 + \frac{1}{2} k_{4\mu}\epsilon_{4\nu}\bar{u}_1 \Gamma^{[\mu\nu]}\not\epsilon_2 u_3\right]. \end{aligned} \tag{3.19}$$

The amplitude with 2 Ramond-Ramond tensors $((NS, \widetilde{NS}) + (R, \widetilde{R}) \to (NS, \widetilde{NS}) + (R, \widetilde{R}))$ is:

$$\mathcal{A}_{RBRB} = -g_{2R}\, K_{FBFB}\, \tilde{K}_{FBFB}\, \frac{\Gamma(-\frac{\alpha' s}{4})\Gamma(-\frac{\alpha' t}{4})\Gamma(-\frac{\alpha' u}{4})}{\Gamma(1+\frac{\alpha' s}{4})\Gamma(1+\frac{\alpha' t}{4})\Gamma(1+\frac{\alpha' u}{4})}\,, \tag{3.20}$$

with

$$g_{2R} \;=\; \frac{\pi^2\, g_c^2}{\alpha'}\,. \tag{3.21}$$

The (right-moving) kinematic factor $\tilde{K}_{FBFB}$ is obtained from K_{FBFB} by putting a tilde on every u and ϵ.

We also need two 3-particle vertices with one massive tensor. The results are:

- Massive boson of $(NS, \widetilde{NS}) \to$ 2 massless fermion $(R, \widetilde{NS}) + (R, \widetilde{NS})$:

$$\begin{aligned} \mathcal{A}_{MFF}(k, \alpha, \sigma, \tilde{\alpha}, \tilde{\sigma}; k_1, u_1, \tilde{\epsilon}_1; k_2, u_2, \tilde{\epsilon}_2) \;=\; & g_{MFF} K_{MFF}(k, \alpha, \sigma; k_1, u_1; k_2, u_2) \times \\ & \times \tilde{K}_M(k, \tilde{\alpha}, \tilde{\sigma}; k_1, \tilde{\epsilon}_1; k_2, \tilde{\epsilon}_2)\,, \end{aligned} \tag{3.22}$$

$$g_{MFF} \;=\; \frac{4\pi\, g_c}{\alpha'}\,, \tag{3.23}$$

where the kinematic factor K_{MFF} is:

$$K_{MFF} = \frac{1}{2}\sqrt{\frac{\alpha'}{2}}\alpha_{\mu\nu\rho}\bar{u}_1\Gamma^{\mu\nu\rho}u_2 - \frac{\alpha'}{2}\,\sigma_{\mu\nu}k_2^\mu\, \bar{u}_1\Gamma^\nu u_2\,. \tag{3.24}$$

- Massive boson $(NS, \widetilde{NS}) \to$ massless tensor $(R, \widetilde{R}) + (R, \widetilde{R})$:

$$\mathcal{A}_{MRR} = -g_{MRR} K_{MFF} \tilde{K}_{MFF}\,, \qquad g_{MRR} = \frac{2\pi\, g_c}{\alpha'}\,. \tag{3.25}$$

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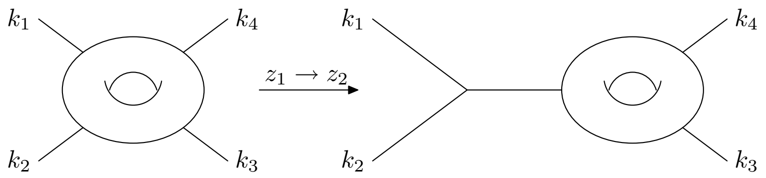

Figure 2: The colliding limit of $z_1 \to z_2$ gives a 3-particle one-loop amplitude.

4. One loop amplitudes and their factorization and unitarity

4.1 One loop amplitudes and their factorization

At one loop, the four particle amplitude was firstly computed by Green and Schwarz in [32] and the result is:

$$\mathcal{A}_4^{1-\text{loop}} = g_4^{1-\text{loop}} \, K(k_i, \epsilon_i) \int_F \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^2} \int \prod_{i=1}^{4} \frac{\mathrm{d}^2 z_i}{\mathrm{Im}\,\tau} \times$$
$$\times \prod_{r<s} \left| \frac{\Theta_1(z_{rs}|\tau)}{\partial\Theta_1(0|\tau)} \exp\left(\frac{-\pi}{\mathrm{Im}\,\tau}(\mathrm{Im}\, z_{rs})^2\right) \right|^{\alpha' k_r \cdot k_s} , \tag{4.1}$$

where $z_{rs} = z_r - z_s$. What we required in this paper is the 3-particle amplitude with one massive tensor at 1-loop. We can obtain it by factorization but we must make an assumption that the kinematic factor is the same as that appearing at tree level. This was explicitly checked by an explicit calculation at one-loop [33]. Now we derive this 3-particle amplitude by using factorization.

By taking the limit of $z_1 \to z_2$ to select the physical pole term as $s = -(k_1+k_2)^2 \to \frac{4}{\alpha'}$, we have:

$$\mathcal{A}_4^{1-\text{loop}} \to g_4^{1-\text{loop}} \, K(k_i, \epsilon_i) \frac{-\frac{4\pi}{\alpha'}}{s - \frac{4}{\alpha'}} \int_F \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^2} \int \frac{\mathrm{d}^2 z_2}{(\mathrm{Im}\,\tau)^2} \prod_{i=3}^{4} \frac{\mathrm{d}^2 z_i}{\mathrm{Im}\,\tau} \times$$
$$\times \prod_{2\le r<s} \left| \frac{\Theta_1(z_{rs}|\tau)}{\partial\Theta_1(0|\tau)} \exp\left(\frac{-\pi}{\mathrm{Im}\,\tau}(\mathrm{Im}\, z_{rs})^2\right) \right|^{\alpha' k'_r \cdot k'_s} , \tag{4.2}$$

where $k'_2 = k_1 + k_2$ and $k'_{3,4} = k_{3,4}$. From this result we can extract the (1-loop correction to the) 3-particle amplitude by using the factorization limit as shown in figure 2. We have:

$$\mathcal{A}_{2BM}^{1-\text{loop}}(k_1, k_2, k) = g_3^{1-\text{loop}} \, K_M \tilde{K}_M \int_F \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^5} \int \prod_{i=1}^{2} \mathrm{d}^2 z_i \times$$
$$\times \prod_{r<s}^{3} \left| \frac{\Theta_1(z_{rs}|\tau)}{\partial\Theta_1(0|\tau)} \exp\left(\frac{-\pi}{\mathrm{Im}\,\tau}(\mathrm{Im}\, z_{rs})^2\right) \right|^{\alpha' k_r \cdot k_s} , \tag{4.3}$$

$$g_4^{1-\text{loop}} = \frac{g_M \, g_3^{1-\text{loop}}}{4\pi/\alpha'} , \tag{4.4}$$

by setting $k_3 = k$. Here we have used the invariance of the integrand under translation of all the insertion points to fix $z = z_3$ to an arbitrary point on the torus (so there is no

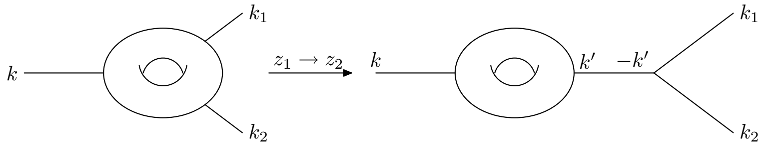

Figure 3: Further degeneration $z_1 \to z_2$ gives the 2-particle one-loop amplitude.

integration over z_3 in eq. (4.3)). Now we can use factorization again to this amplitude by taking the limit $z_1 \to z_2$ to select the physical pole term as $s = -(k_1+k_2)^2 \to \frac{4}{\alpha'}$. We have:

$$\mathcal{A}_{2BM}^{1-\text{loop}} \to g_3^{1-\text{loop}} K_M \tilde{K}_M \frac{-\frac{4\pi}{\alpha'}}{s-\frac{4}{\alpha'}} \int_F \frac{\mathrm{d}^2\tau}{(\operatorname{Im}\tau)^5} \int \mathrm{d}^2 z_2 \times$$
$$\times \left| \frac{\Theta_1(z_2|\tau)}{\partial\Theta_1(0|\tau)} \exp\left(\frac{-\pi}{\operatorname{Im}\tau}(\operatorname{Im} z_2)^2\right)\right|^{\alpha'(k_1+k_2)\cdot k} . \tag{4.5}$$

From this result we can extract the (1-loop correction to the) 2-particle amplitude by using the factorization limit as shown in figure 3. We have:

$$\mathcal{A}_{MM}^{1-\text{loop}}(k,k') = g_{MM}^{1-\text{loop}} K_{MM}\tilde{K}_{MM} \int_F \frac{\mathrm{d}^2\tau}{(\operatorname{Im}\tau)^5} \int \mathrm{d}^2 z \times$$
$$\times \left| \frac{\Theta_1(z|\tau)}{\partial\Theta_1(0|\tau)} \exp\left(\frac{-\pi}{\operatorname{Im}\tau}(\operatorname{Im} z)^2\right)\right|^{\alpha' k\cdot k'} , \tag{4.6}$$

$$g_3^{1-\text{loop}} = \frac{g_M\, g_{MM}^{1-\text{loop}}}{4\pi\alpha'} , \tag{4.7}$$

where the kinematic factor K_{MM} is:

$$K_{MM} = -6\alpha_{\mu_1\nu_1\rho_1}(k)\alpha^{\mu_1\nu_1\rho_1}(k') + \sigma_{\mu\nu}(k)\sigma^{\mu\nu}(k') . \tag{4.8}$$

4.2 The one-loop unitarity relation for the massive tensor propagator

We will use the following formula from the operator formalism of string theory:

$$\operatorname{Tr}(V(k_1, \mathrm{e}^{2\pi i z_1})V(k_2, \mathrm{e}^{2\pi i z_2}) \cdots V(k_M, \mathrm{e}^{2\pi i z_M})\, q^N\, \bar{q}^{\tilde{N}} (q\bar{q})^{\frac{\alpha'}{4}\hat{p}^2}) =$$
$$= i \left[\frac{-2\pi}{\alpha' \ln|q|}\right]^{D/2} \prod_{r<s} \left| 2\pi \frac{\Theta_1(z_{rs}|\tau)}{\partial\Theta_1(0|\tau)} \exp\left(\frac{-\pi}{\operatorname{Im}\tau}(\operatorname{Im} z_{rs})^2\right)\right|^{\alpha' k_r\cdot k_s} , \tag{4.9}$$

where $q = \mathrm{e}^{2\pi i z_M} = \mathrm{e}^{2\pi i\tau}$ and $\mathrm{Imz_1} \le \mathrm{Imz_2} \le \cdots \le \mathrm{Imz_M} = \operatorname{Im}\tau$. The vertex operator V is:

$$V(k, \mathrm{e}^{2\pi i z}) = \mathrm{e}^{\ \frac{\alpha'}{2}\ ^{1/2} \sum_{n=1}^{\infty} \frac{1}{n}\left(k\cdot\alpha_{-n}\,\mathrm{e}^{2\pi i n z} + k\cdot\tilde{\alpha}_{-n}\,\mathrm{e}^{2\pi i n \bar{z}}\right)} \mathrm{e}^{ik\cdot(\hat{x}+i2\pi\alpha' \operatorname{Im} z\hat{p})} \times$$
$$\times \mathrm{e}^{-\ \frac{\alpha'}{2}\ ^{1/2} \sum_{n=1}^{\infty} \frac{1}{n}\left(k\cdot\alpha_{n}\,\mathrm{e}^{-2\pi i n z} + k\cdot\tilde{\alpha}_{n}\,\mathrm{e}^{-2\pi i n \bar{z}}\right)} , \tag{4.10}$$

and $V_R(k) = V(k, \mathrm{e}^{2\pi i z})|_{z\to 0, \hat{x}\to 0}$ which appears soon.

The trace contains an integration over momentum and summations over all the Fock states created by the creation operators α_{-n} and $\tilde{\alpha}_{-n}$ and their super-partners (omitting the zero modes) (which are required to cancel the factor $|\eta(\tau)|^D$ in bosonic string theory).

For $M = 2$ after evaluating the trace by inserting a complete set of intermediate states, we have

$$\begin{aligned}\frac{i(2\pi)^{\alpha' k\cdot k'}}{(\alpha' \mathrm{Im}\,\tau)^{D/2}} &\left|\frac{\Theta_1(\tau - z|\tau)}{\partial\Theta_1(0|\tau)} \exp\left(\frac{-\pi}{\mathrm{Im}\tau}(\mathrm{Im}(\tau - z))^2\right)\right|^{\alpha' k\cdot k'} = \\ &= \int \mathrm{d}^D p \left(\mathrm{e}^{-4\pi\,\mathrm{Im}\,z}\right)^{\frac{\alpha'}{4}p^2} \left(\mathrm{e}^{-4\pi\,\mathrm{Im}(\tau - z)}\right)^{\frac{\alpha'}{4}(p+k')^2} \times \\ &\times \sum_{n,\tilde{n},m,\tilde{m}} \mathrm{e}^{2\pi i(n\,z - \tilde{n}\bar{z})} \langle n, \tilde{n}|V_R(k)|m, \tilde{m}\rangle \times \\ &\times \langle m, \tilde{m}|V_R(k')|n, \tilde{n}\rangle\, \mathrm{e}^{2\pi i(m\,(\tau - z) - \tilde{m}(\bar{\tau} - \bar{z}))}\,. \end{aligned} \tag{4.11}$$

Now we can do the integration over τ and z explicitly. We introduce an ultraviolet cutoff Λ by restricting the integration to the following region:

$$\mathrm{Im}\,\tau \geq 2\Lambda, \qquad \Lambda \leq \mathrm{Im}\,z \leq \mathrm{Im}\,\tau - \Lambda\,. \tag{4.12}$$

After carrying out the integration explicitly we have

$$\begin{aligned}\frac{i(2\pi)^{\alpha' k\cdot k'}}{(\alpha')^{D/2}} \int_{\mathrm{Im}\,\tau\geq\Lambda} &\frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^{D/2}} \int' \mathrm{d}^2 z \left|\frac{\Theta_1(\tau - z|\tau)}{\partial\Theta_1(0|\tau)} \exp\left(\frac{-\pi}{\mathrm{Im}\tau}(\mathrm{Im}(\tau - z))^2\right)\right|^{\alpha' k\cdot k'} = \\ = \int \frac{\mathrm{d}^D p}{(\pi\alpha')^2} &\sum_{n,\tilde{n},m,\tilde{m}} \frac{\mathrm{e}^{-\Lambda(p^2 + \frac{4}{\alpha'}n - i\epsilon) - \Lambda((p-k)^2 + \frac{4}{\alpha'}m - i\epsilon)}}{(p^2 + \frac{4}{\alpha'}\,n - i\epsilon)((p-k)^2 + \frac{4}{\alpha'}m - i\epsilon)} \times \\ \times \delta_{n,\tilde{n}}\delta_{m,\tilde{m}} &\langle n, \tilde{n}|V_R(k)|m, \tilde{m}\rangle \langle m, \tilde{m}|V_R(k')|n, \tilde{n}\rangle\,. \end{aligned} \tag{4.13}$$

By assuming $s = -k^2 = k\cdot k' \to \frac{4}{\alpha'}$, we see that only the "ground state" could contribute an imaginary part in the above summation (see below in eq. (4.15)), and we have:

$$\begin{aligned} i\mathcal{A}^{\text{1-loop}}_{MM} = g^{\text{1-loop}}_{MM} K_{MM}\tilde{K}_{MM} &\frac{(\alpha')^{D/2}}{(2\pi)^{\alpha' k\cdot k'}} \times \\ \times \int \frac{\mathrm{d}^D p}{(\pi\alpha')^2} &\frac{\mathrm{e}^{-\Lambda(p^2 + (p-k)^2 - i\epsilon)}}{(p^2 - i\epsilon)((p-k)^2 - i\epsilon)} + i\,(\text{Purely real parts})\,. \end{aligned} \tag{4.14}$$

From field theory or by explicit calculation, we have:

$$\begin{aligned} iA(s + i\epsilon) &= \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{p^2 + m^2 - i\epsilon} \frac{1}{(p-k)^2 + m^2 - i\epsilon} \\ \mathrm{Disc}A(s) &= \Theta(s - 4m^2)\,\frac{iS_{D-1}}{(2\pi)^{D-2}} \frac{\left(\frac{s}{4} - m^2\right)^{\frac{D-3}{2}}}{4s^{1/2}} \\ &= i\int \frac{\mathrm{d^D} k_1}{(2\pi)^D}\, 2\pi\delta(k_1^2 + m^2) \int \frac{\mathrm{d^D} k_2}{(2\pi)^D}\, 2\pi\delta(k_2^2 + m_2^2) \times \\ &\quad \times (2\pi)^D \delta^D(k_1 + k_2 + k)\,|A^{\text{tree}}(k_1; k_2; k)|^2\,. \end{aligned} \tag{4.15}$$

where $s = -k^2$ and $\mathrm{Disc}A(s) \equiv A(s + i\epsilon) - A(s - i\epsilon)$.

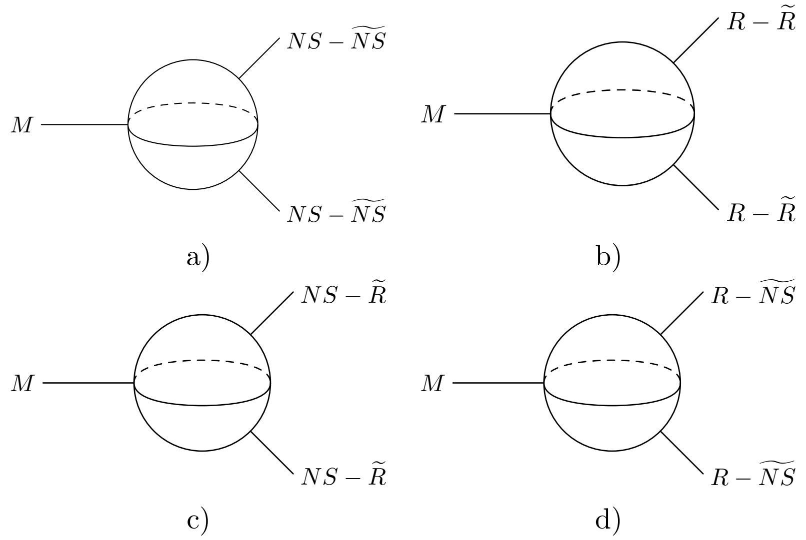

Figure 4: The 4 possible contributions to $\mathcal{A}^{\rm tree}_{M**}$.

The unitarity relation is:

$$\begin{aligned}\mathcal{A}(s+i\epsilon)-\mathcal{A}(s-i\epsilon) &= \frac{i}{2!}\int\frac{\mathrm{d}^{\rm D}k_1}{(2\pi)^D}\,2\pi\delta(k_1^2)\int\frac{\mathrm{d}^{\rm D}k_2}{(2\pi)^D}\,2\pi\delta(k_2^2)\,\times\\ &\quad\times(2\pi)^D\delta^D(k_1+k_2+k)\,|\mathcal{A}^{\rm tree}_{M**}(k_1;k_2;k)|^2\,,\end{aligned}\tag{4.16}$$

where the factor of 2 is due to the propagation of identical particles.

For superstring theory in ten dimensions, the possible $\mathcal{A}^{\rm tree}_{M**}$'s are listed in figure 4. In appendix we will prove the following result for the summation over all possible intermediate states:

$$\sum_{\text{all intermediate states}}|\mathcal{A}^{\rm tree}_{M**}|^2=(g_{_M})^2\,K_{MM}\tilde{K}_{MM}\,.\tag{4.17}$$

We relegate the proof of this result to appendix B.

By using this result we have:

$$g^{1-{\rm loop}}_{MM}\,\frac{(\alpha')^{D/2}}{(2\pi)^{\alpha' k\cdot k'}}\,\frac{(2\pi)^D}{(\pi\alpha')^2}=\frac{(g_{_M})^2}{2}\,.\tag{4.18}$$

by using this equation with eqs. (4.4) and (4.7) we have:

$$g^{1-{\rm loop}}_{_M}=\frac{g_c^2}{2\pi^2(\alpha')^5}\,,\tag{4.19}$$

$$g^{1-{\rm loop}}_{3}=\frac{g_c^3}{\pi^2(\alpha')^5}\,,\tag{4.20}$$

$$g^{1-{\rm loop}}_{4}=\frac{2g_c^3}{\pi^2(\alpha')^5}\,.\tag{4.21}$$

The coefficient $g_4^{1-{\rm loop}}$ agrees with the result of Sakai and Tanni [34].

5. The factorization of the two loop four particle amplitude

In this section we will use the result of [22] to determine the overall coefficient of the two-loop 4-particle amplitude. We pay particular attention to the overall coefficient. To begin with, let us recall the two-loop four-particle amplitude in type-II superstring theories obtained in refs. [2, 3]:

$$\begin{aligned}\mathcal{A}_{II} &= C_{II} K(k_i,\epsilon_i)\,\frac{1}{6!}\int \frac{1}{T^5}\frac{\prod_{i=1}^{6}\mathrm{d}^2a_i}{\mathrm{d}V_{pr}|\prod_{i<j}a_{ij}|^2}\times\\ &\times\prod_{i=1}^{4}\frac{\mathrm{d}^2z_i}{|y(z_i)|^2}\prod_{i<j}\exp\{-k_i\cdot k_j\langle X(z_i)X(z_j)\rangle\},\\ &\times|s(z_1z_2+z_3z_4)+t(z_1z_4+z_2z_3)+u(z_1z_3+z_2z_4)|^2\,, \end{aligned}\tag{5.1}$$

where

$$\mathrm{d}V_{pr} = \frac{\mathrm{d}^2a_i\mathrm{d}^2a_j\mathrm{d}^2a_k}{|a_{ij}a_{ik}a_{jk}|^2}\,,\tag{5.2}$$

$$T = \int \mathrm{d}^2z_1\mathrm{d}^2z_2\frac{|z_1-z_2|^2}{|y(z_1)y(z_2)|^2}\,,\tag{5.3}$$

$$y^2(z) = \prod_{i=1}^{6}(z-a_i)\,,\tag{5.4}$$

and $\langle X(z_i)X(z_j)\rangle \equiv \langle X(z_i,\bar z_i)X(z_j,\bar z_j)\rangle$'s are the scalar correlators. The $K(k_i,\epsilon_i)$ is the standard kinematic factor appearing at tree, one- and two-loop computations [32, 1, 3]. C_{II} is an overall factor which will be determined in this section.

There are 10 possible ways for the dividing degeneration limit (one is $a_2-a_1=u$, $a_3-a_1=vu$ and $u\to 0$)[3] and by using the result of [22] we have:

$$\begin{aligned}\mathcal{A}_{II} \to{}& C_{II}\,K(k_i,\epsilon_i)\,\frac{10}{6!}\,\frac{\frac{2\pi}{(\alpha')^3}}{-s+\frac{4}{\alpha'}}\int\frac{|K_1K_2/4|^2}{T_1^5T_2^5}\frac{\mathrm{d}^2a_1}{|a_{14}a_{15}a_{16}|^2}\frac{\mathrm{d}^2v}{|v(v-1)|^2}\times\\ &\times\frac{\mathrm{d}^2x_1\mathrm{d}^2x_2}{|y_1(x_1)y_1(x_2)|^2}\frac{\mathrm{d}^2z_3\mathrm{d}^2z_4}{|y_2(z_3)y_2(z_4)|^2}\\ &\times\exp\Big\{-\Big(G_1(x_1,x_2)-G_1(x_1,p_1)-G_1(x_2,p_1)\Big)-\\ &\qquad-\Big(G_2(z_3,z_4)-G_2(z_3,p_2)-G_2(z_4,p_2)\Big)\Big\}\,.\end{aligned}\tag{5.5}$$

The one-loop amplitude in hyperelliptic language is:

$$\begin{aligned}&\frac{1}{4!}\int\frac{|K|^2}{T^5}\frac{\mathrm{d}^2a_1}{|a_{12}a_{13}a_{14}|^2}\frac{\mathrm{d}^2z_1\mathrm{d}^2z_2}{|y(z_1)y(z_2)|^2}\exp\Big\{-\Big(G(z_1,z_2)-G(z_1,z_3)-G(z_2,z_3)\Big)\Big\}=\\ &\quad=\frac{1}{(2\pi)^2}\int_F\frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^5}\int\prod_{i=1}^{2}\mathrm{d}^2z_i\prod_{r<s}^{3}\left|\frac{\Theta_1(z_{rs}|\tau)}{\partial\Theta_1(0|\tau)}\exp\left(\frac{-\pi}{\mathrm{Im}\tau}(\mathrm{Im}\,z_{rs})^2\right)\right|^{\alpha' k_r\cdot k_s}\,.\end{aligned}\tag{5.6}$$

[3] There are some fine points which should be taken into account. See section 3.2 of [27] for details.

This gives the following relation by using factorization relation:

$$C_{II}\,\frac{10}{6!}\times\frac{\pi}{2^3(\alpha')^3}\times\left(\frac{4!}{(2\pi)^2}\right)^2=(g_3^{1-\mathrm{loop}})^2\,. \tag{5.7}$$

This gives

$$C_{II}=\frac{2^4\,g_c^6}{(\alpha')^7\pi}\,. \tag{5.8}$$

In period matrix language we have:[4]

$$\begin{aligned}\mathcal{A}_{II}=C_{II}\,\frac{1}{(2\pi)^6\,2^5}\,K(k_i,\epsilon_i)\int\frac{|\mathrm{d}^3\tau|^2}{(\det\mathrm{Im})^5}\times\\ \times\int\prod_{i=1}^4\mathrm{d}^2z_i|3\mathcal{Y}_s|^2\prod_{i<j}\exp\{-k_i\cdot k_j\langle X(z_i)X(z_j)\rangle\}\,,\end{aligned} \tag{5.9}$$

and the overall coefficient is

$$\hat{C}_{II}=C_{II}\,\frac{1}{(2\pi)^6\,2^5}=\frac{g_c^6}{(2\pi\alpha')^7}\,. \tag{5.10}$$

This result agrees with D'Hoker, Gutperle and Phong [27] by taking into account the different convention for d^2z (we use $\mathrm{d}^2z=\mathrm{d}x\mathrm{d}y$ for $z=x+iy$).

A. Formulas for tensor integration

Here is a list of all the formulas needed for tensor integrations which are used in appendix B to prove eq. (4.17). We have

$$\int\frac{\mathrm{d}^Dp}{(2\pi)^D}\,p^\mu\delta(p^2)\,\delta((p+k)^2)=-\frac{1}{2}\,k^\mu\int\frac{\mathrm{d}^Dp}{(2\pi)^D}\,\delta(p^2)\,\delta((p+k)^2)\,. \tag{A.1}$$

$$\begin{aligned}\int\frac{\mathrm{d}^Dp}{(2\pi)^D}\,p^{\mu_1}p^{\mu_2}\delta(p^2)\,\delta((p+k)^2)&=\frac{1}{4(D-1)}(-k^2\,\eta^{\mu_1\mu_2}+D\,k^{\mu_1}k^{\mu_2})\times\\ &\quad\times\int\frac{\mathrm{d}^Dp}{(2\pi)^D}\,\delta(p^2)\,\delta((p+k)^2)\,.\end{aligned} \tag{A.2}$$

$$\begin{aligned}\int\frac{\mathrm{d}^Dp}{(2\pi)^D}\,p^{\mu_1}p^{\mu_2}p^{\mu_3}p^{\mu_4}\delta(p^2)\,\delta((p+k)^2)&=\frac{1}{16(D^2-1)}\times\\ &\quad\times\Big(k^4(\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}+(\mu_1\leftrightarrow\mu_3)+(\mu_1\leftrightarrow\mu_4))-\\ &\qquad-k^2(D{+}2)(k^{\mu_1}k^{\mu_2}g^{\mu_3\mu_4}+(5\text{ more terms}))+\\ &\qquad+(D+2)(D+4)k^{\mu_1}k^{\mu_2}k^{\mu_3}k^{\mu_4}\Big)\times\\ &\quad\times\int\frac{\mathrm{d}^Dp}{(2\pi)^D}\,\delta(p^2)\,\delta((p+k)^2)\,.\end{aligned} \tag{A.3}$$

$$\int\frac{\mathrm{d}^Dp}{(2\pi)^D}\,p^{\mu_1}p^{\mu_2}p^{\mu_3}p^{\mu_4}p^{\mu_5}p^{\mu_6}\delta(p^2)\,\delta((p+k)^2)=$$

[4] The factor $\frac{\pi}{2}$ in the last equation of [3] should be 2π.

$$
\begin{aligned}
&= \frac{1}{64(D+3)(D^2-1)} \times \\
&\quad \times \Big(- k^6 (\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}\eta^{\mu_5\mu_6} + (14 \text{ more terms})) + \\
&\quad\quad + k^4 (D+4)(k^{\mu_1}k^{\mu_2}\eta^{\mu_3\mu_4}\eta^{\mu_5\mu_6} + \\
&\quad\quad\quad\quad + (44 \text{ more terms})) \\
&\quad\quad - k^2 (D+4)(D+6) \times \\
&\quad\quad \times (k^{\mu_1}k^{\mu_2}k^{\mu_3}k^{\mu_4}\eta^{\mu_5\mu_6} + (14 \text{ more terms})) + \\
&\quad\quad + (D+4)(D+6)(D+8) \times \\
&\quad\quad \times k^{\mu_1}k^{\mu_2}k^{\mu_3}k^{\mu_4}k^{\mu_5}k^{\mu_6} \Big) \\
&\quad \times \int \frac{\mathrm{d}^D p}{(2\pi)^D}\, \delta(p^2)\, \delta((p+k)^2)\,. \qquad \text{(A.4)}
\end{aligned}
$$

$$
\begin{aligned}
&\int \frac{\mathrm{d}^D p}{(2\pi)^D}\, p^{\mu_1}p^{\mu_2}p^{\mu_3}p^{\mu_4}p^{\mu_5}p^{\mu_6}p^{\mu_7}p^{\mu_8}\delta(p^2)\,\delta((p+k)^2) = \\
&= \frac{1}{256(D+5)(D+3)(D^2-1)} \times \\
&\quad \times \Big(k^8 (\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}\eta^{\mu_5\mu_6}\eta^{\mu_7\mu_8} + \\
&\quad\quad + (104 \text{ more terms})) - \\
&\quad\quad - k^6 (D+6)(k^{\mu_1}k^{\mu_2}\eta^{\mu_3\mu_4}\eta^{\mu_5\mu_6}\eta^{\mu_7\mu_8} + \\
&\quad\quad\quad\quad + (419 \text{ more terms})) + \\
&\quad\quad + k^4 (D+6)(D+8) \times \\
&\quad\quad \times (k^{\mu_1}k^{\mu_2}k^{\mu_3}k^{\mu_4}\eta^{\mu_5\mu_6}\eta^{\mu_7\mu_8} + (209 \text{ more})) - \\
&\quad\quad - k^2 (D+6)(D+8)(D+10) \times \\
&\quad\quad \times (k^{\mu_1}k^{\mu_2}k^{\mu_3}k^{\mu_4}k^{\mu_5}k^{\mu_6}\eta^{\mu_7\mu_8} + (27 \text{ more})) + \\
&\quad\quad + (D+6)(D+8)(D+10)(D+12) \times \\
&\quad\quad \times k^{\mu_1}k^{\mu_2}k^{\mu_3}k^{\mu_4}k^{\mu_5}k^{\mu_6}k^{\mu_7}k^{\mu_8} \Big) \times \\
&\quad \times \int \frac{\mathrm{d}^D p}{(2\pi)^D}\, \delta(p^2)\, \delta((p+k)^2)\,. \qquad \text{(A.5)}
\end{aligned}
$$

B. Proof of eq. (4.17): the summation over the intermediate states in one-loop unitarity relation

As we explained in subsection 4.2, in order to use the 1-loop unitarity relation to determine the coefficient $g_M^{1-\text{loop}}$, we need eq. (4.17) to do the summation over all the intermediate states. This equation is only true when it is inserted in an integral where we can use the substitution rules implied by the formulas for tensor integrations derived in the last appendix. Here we will give some details for the summation over all the intermediate states and sketch a proof of eq. (4.17).

The summation over the intermediate states can be done separately for the left-moving part and the right-moving part. So we need the formulas for the NS and R intermediate states. For NS intermediate states, the kinematic factor K_M is given in eq. (3.9) and we have:

$$\sum_{\epsilon_1,\epsilon_2} |K_M(p_1,\epsilon_1;p_2,\epsilon_2;k,\alpha,\sigma)|^2 = -36\alpha_{\mu_1\nu\rho}(k)\alpha_{\mu_2}{}^{\nu\rho}(-k)p_1^{\mu_1}p_1^{\mu_2} + \sigma_{\mu_1\nu_1}(k)\sigma_{\mu_2\nu_2}(-k)\times$$
$$\times\Big\{(D-2)p_1^{\mu_1}p_1^{\mu_2}p_1^{\nu_1}p_1^{\nu_2} - 4p_1^{\mu_1}p_1^{\mu_2}\eta^{\nu_1\nu_2} + \eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2}\Big\}, \tag{B.1}$$

by using the following summation formula:

$$\sum_{\epsilon}\epsilon^\mu(p)\epsilon^\nu(-p) = \eta^{\mu\nu} - \frac{p^\mu p'^\nu}{2|p|^2} - \frac{p'^\mu p^\nu}{2|p|^2}, \tag{B.2}$$

where $p=(p^0,p^1,p^2,p^3)$ and $p'=(-p^0,p^1,p^2,p^3)$. D is the dimension of space-time.

For R intermediate states, the kinematic factor K_{MFF} is given in eq. (3.24) and we have:

$$\sum_{u_1,u_2}|K_{MFF}(k,\alpha,\sigma;p_1,u_1;p_2,u_2;)|^2 = \alpha_{\mu_1\nu_1\rho_1}(k)\alpha_{\mu_2\nu_2\rho_2}(-k)\,A^{\mu_1\nu_1\rho_1\mu_2\nu_2\rho_2} -$$
$$-\frac{N}{2}\sigma_{\mu_1\nu_1}(k)\sigma_{\mu_2\nu_2}(-k)\times$$
$$\times(p_1^{\mu_1}p_1^{\mu_2}p_1^{\nu_1}p_1^{\nu_2} + \frac{k^2}{4}\eta^{\nu_1\nu_2}\,p_1^{\mu_1}p_1^{\mu_2}), \tag{B.3}$$

where $N=32$ is the dimension of Dirac spinor and

$$\begin{aligned} A^{\mu_1\nu_1\rho_1\mu_2\nu_2\rho_2} &= \frac{1}{8}\,\mathrm{Tr}\left[\not{p}_1\Gamma^{[\mu_1\nu_1\rho_1]}\not{p}_2\Gamma^{[\mu_2\nu_2\rho_2]}\frac{1+\Gamma^{11}}{2}\right] \\ &= \frac{1}{16}\,\mathrm{Tr}\left[\not{p}_1\Gamma^{[\mu_1\nu_1\rho_1]}\not{p}_2\Gamma^{[\mu_2\nu_2\rho_2]}\right]. \end{aligned} \tag{B.4}$$

Here we have used:

$$\sum_u u(p)\,\bar{u}(-p) = \frac{1+\Gamma^{11}}{2}\not{p}. \tag{B.5}$$

The 4 different contributions as displayed in figure 4 are given as follows:

$$\begin{aligned} \mathcal{A}_{M2(NS-\widetilde{NS})} &= g_{_M}\,K_M(k_1,\epsilon_1;k_2,\epsilon_2;k,\alpha,\sigma)\times \\ &\quad\times K_M(k_1,\tilde{\epsilon}_1;k_2,\tilde{\epsilon}_2;k,\tilde{\alpha},\tilde{\sigma}), \end{aligned} \tag{B.6}$$

$$\begin{aligned} \mathcal{A}_{M2(R-\tilde{R})} &= -g_{_{MRR}}\,K_{MFF}(k_1,u_1;k_2,u_2;k,\alpha,\sigma)\times \\ &\quad\times K_{MFF}(k_1,\tilde{u}_1;k_2,\tilde{u}_2;k,\tilde{\alpha},\tilde{\sigma}), \end{aligned} \tag{B.7}$$

$$\begin{aligned} \mathcal{A}_{M2(R-\widetilde{NS})} &= g_{_{MFF}}\,K_{MFF}(k_1,u_1;k_2,u_2;k,\alpha,\sigma)\times \\ &\quad\times K_M(k_1,\tilde{\epsilon}_1;k_2,\tilde{\epsilon}_2;k,\tilde{\alpha},\tilde{\sigma}), \end{aligned} \tag{B.8}$$

$$\begin{aligned} \mathcal{A}_{M2(NS-\tilde{R})} &= g_{_{MFF}}\,K_{MFF}(k_1,u_1;k_2,u_2;k,\alpha,\sigma)\times \\ &\quad\times K_M(k_1,\tilde{\epsilon}_1;k_2,\tilde{\epsilon}_2;k,\tilde{\alpha},\tilde{\sigma}). \end{aligned} \tag{B.9}$$

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By using these results we have

$$\sum_{\text{all intermediate states}} |\mathcal{A}_{M**}^{\text{tree}}|^2 = \sum_{\epsilon_i,\tilde{\epsilon}_i} \left|\mathcal{A}_{M2(NS-\widetilde{NS})}\right|^2 + \sum_{u_i,\tilde{u}_i} \left|\mathcal{A}_{M2(R-\tilde{R})}\right|^2 + \\ + \sum_{\epsilon_i,\tilde{u}_i} \left|\mathcal{A}_{M2(NS-\tilde{R})}\right|^2 + \sum_{u_i,\tilde{\epsilon}_i} \left|\mathcal{A}_{M2(R-\widetilde{NS})}\right|^2 . \qquad \text{(B.10)}$$

Now we can use eqs. (B.1) and (B.3) to do the summation over the intermediates. The results can be simplified further by using the formulas in appendix A for the integration over k_1 ($k_2 = -(k+k_1)$ by momentum conservation). After a long and tedious calculation, eq. (4.17) is proved.

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